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FUNCTIONAL ANALYSIS

On Series Whose Permutations Have Only Two Sums

by

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Summary. The purpose of this work is to show that in every infinite dimensional Banach space X there exists a sequence (x_n), $x_n \in X$, such that $\text{card}\{x \in X: \exists_{\pi:N\to N},\ x = \sum x_{\pi(n)}\} = 2$, where π is a permutation of set of integers.

Introduction. Let $\sum x_n$ be a convergent series in a Banach space X, such that $\sum \|x_n\| = \infty$. Let us denote by $OC(x_n)$ the set

$$\{x \in X: \exists_{\pi:N\to N},\ x = \sum x_{\pi(n)}\},$$

where π is the permutation of the set of integers. At the beginning of this century P. Levy [1] and E. Steinitz [2] have shown that if X is finite dimensional then $OC(x_n)$ is linear i. e. $OC(x_n) = v + H$ where $v \in X$ and H is a subspace of X. In "Scottish Book" S. Banach asked the question: "Does this theorem hold in any Banach space X?" The answer is "no" and J. Marcinkiewicz has given a simple counterexample [6]. Independently, the Russian mathematicians investigating convergence in Banach spaces have constructed another series, for which $OC(\cdot)$ is not linear. However, always the sets $OC(\cdot)$ were infinite, for example the algebraic groups. We want to show that nonlinear $OC(\cdot)$ can be as small as possible, i.e. for some sequence (x_n) $\text{card}(OC(x_n)) = 2$.

First, we establish the notation. By $L_p(Q)$ we will denote $L_p(Q, \mathfrak{B}, \lambda)$ where $Q = [0, 1]^\omega$, $\mathfrak{B}$ is the σ-ring of Borel subsets of Q and λ—the standard probability measure on $\mathfrak{B}$. Measurable function on Q, which equals $c (c \in \mathbf{R})$ will be denoted by c. The greek letters π, σ will always denote permutations of the set of natural numbers. Now we can formulate our main technical result:

PROPOSITION. *There exists a sequence $(h_n)_{n=1}^{\infty}$, $h_n \in L_\infty(Q)$ such that*

(i) $\sum_{n=1}^{\infty} h_n = 0$ *in* $L_p(Q)$ $1 \leqslant p < \infty$ *and* $\|\sum_{n=1}^{N} h_n\|_p \leqslant C_0 N^{-1/3p}$

(ii) *there exists a permutation* π *such that*

$$\sum_{n=1}^{\infty} h_{\pi(n)} = 1 \quad \text{in } L_p(Q)\ 1 \leqslant p < \infty \quad \text{and } \Big\| \sum_{n=1}^{N} h_{\pi(n)} - 1 \Big\|_p \leqslant C_1 N^{-1/3p}$$

(iii) *if* $h_0 \in L_p(Q)$ *and* $h_0 = \sum_{n=1}^{\infty} h_{\sigma(n)}$ *for some permutation* σ *then* $h_0 = 0$ *or* $h_0 = 1$.

THEOREM. *In every infinite dimensional Banach space* X *there exists a sequence* $(x_n)_{n=1}^{\infty}$, $x_n \in X$ *such that*

(i)
$$\sum_{n=1}^{\infty} x_n = y_0$$

(ii) *there exists a permutation* π *such that*

$$\sum_{n=1}^{\infty} x_{\pi(n)} = y_1 \text{ and } y_1 \neq y_0$$

(iii) *if there exists a permutation* σ *such that*

$$\sum_{n=1}^{\infty} x_{\sigma(n)} = y \text{ then } y = y_0 \text{ or } y = y_1.$$

Proof of the Proposition. First we define functions, that form the sequence $(h_n)_{n=1}^{\infty}$. Let

$$f_m^n(t) = f_m^n(t_1, t_2, t_3, \ldots) = \begin{cases} 1 & \text{for } \dfrac{m-1}{n} < t_n < \dfrac{m}{n} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $n \in \mathbf{N}$, $m \in \{1, \ldots, n\}$,

$$g_{m,j}^n = -f_m^n \cdot f_j^{n+1} \quad \text{for } n \in \mathbf{N},\ m \in \{1, \ldots, n\},\ j \in \{1, \ldots, n+1\}. \tag{2}$$

The equalities below are obvious:

$$\sum_{m=1}^{n} f_m^n = 1 = -\sum_{m=1}^{n} \sum_{j=1}^{n+1} g_{m,j}^n \tag{3}$$

$$f_m^n = -\sum_{j=1}^{n+1} g_{m,j}^n \tag{4}$$

$$f_j^{n+1} = -\sum_{m=1}^{n} g_{m,j}^n. \tag{5}$$

Let us consider two series:

$$f_1^1 + g_{1,1}^1 + g_{1,2}^1 + f_1^2 + g_{1,1}^2 + g_{1,2}^2 + g_{1,3}^2 + f_2^2 + g_{2,1}^2 + g_{2,2}^2 + g_{2,3}^2 \ldots$$

$$f_1^1 + f_1^2 + g_{1,1}^1 + f_2^2 + g_{1,2}^1 + f_1^3 + g_{1,1}^2 + g_{2,1}^2 + f_2^3 + g_{1,2}^2 + g_{2,2}^2 + f_3^3 + g_{1,3}^2 + g_{2,3}^2 \ldots$$

We write first series as $\sum_{n=1}^{\infty} h_n$. Then, the second can be written as $\sum_{n=1}^{\infty} h_{\pi(n)}$ for some permutation π.

Let us observe that if $h_N = f_m^n$ or $h_N = g_{m,j}^n$, then $n \sim N^{1/3}$ and if $h_{\pi(N)} = f_m^n$ or $h_{\pi(N)} = g_{m,j}^n$ then $n \sim N^{1/3}$.

From (4) we have

$$\sum_{n=1}^{N} h_n = f_m^n + \sum_{j=1}^{j(N)} g_{m,j}^n \quad \text{where } g_{m,j(N)}^n = h_N$$

or

$$\sum_{n=1}^{N} h_n = f_m^n \quad \text{if } f_m^n = h_N,$$

but

$$\|f_m^n + \sum_{j=1}^{j(N)} g_{m,j}^n\|_p \leqslant \|f_m^n\|_p = 1/n^{1/p} \sim N^{-1/3p}$$

This proves (i). Similarly, using (5) and the fact that $f_1^1 = 1$ we obtain (ii).

Proof of (iii): Since $L_p \subseteq L_1$ for any $p \geqslant 1$ it is sufficient to prove (iii) for $p = 1$. From now on $\|\cdot\|_1$ will be denoted $\|\cdot\|$.

First of all, let us observe that if $h_0 = \sum_{n=1}^{\infty} h_{\sigma(n)}$ in $L_1(Q)$, (4) and (5) (or simpler (3)) and definitions of f_m^n, $g_{m,j}^n$ imply that h does not depend on k-th coordinate for every $k \in \mathbf{N}$. In this case the Kolmogorov Zero-One Law [5] says that h_0 must be a constant function. Thus we can write

(6) $$h_0 = s, \; s \in \mathbf{N}.$$

For the further proof we will need the following lemma:

LEMMA 1. *Let $(X, \mathfrak{X}, \mu)$ and $(Y, \mathfrak{Y}, \nu)$ be probability measure spaces. Let $f, g\colon X \times Y \to \mathbf{R}$ be measurable and integrable functions such that*

$$f(x, y) = \tilde{f}(x), \quad g(x, y) = \tilde{g}(y)$$

then $\|f+g\| \geqslant \|f\| + \|g\|[1 - 2(\mu \times \nu)(\operatorname{supp} f)]$.

Proof of Lemma 1. We have

$$\begin{aligned}
\|f+g\| &= \int_Y \int_X |f(x, y) + g(x, y)| \mu dx \nu dy = \int_Y \int_X |\tilde{f}(x) + \tilde{g}(y)| \mu dx \nu dy \\
&= \int_Y \Big[\int_{X - \operatorname{supp}\tilde{f}} |\tilde{f}(x) + \tilde{g}(y)| \mu dx + \int_{\operatorname{supp}\tilde{f}} |\tilde{f}(x) + \tilde{g}(y)| \mu dx \Big] \nu dy \\
&\geqslant \int_Y \Big[\int_{X - \operatorname{supp}\tilde{f}} |\tilde{g}(y)| \mu dx + \int_{\operatorname{supp}\tilde{f}} |\tilde{f}(x)| - |\tilde{g}(y)| \mu dx \Big] \nu dy \\
&= \int_Y [\|\tilde{f}\| + \|\tilde{g}(y)\| (1 - 2\mu(\operatorname{supp}\tilde{f}))] \nu dy = \|\tilde{f}\| + \|\tilde{g}\| (1 - 2\mu(\operatorname{supp}\tilde{f})).
\end{aligned}$$

Obviously: $\mu(\operatorname{supp}\tilde{f}) = \mu \times \nu(\operatorname{supp} f)$, $\|\tilde{f}\|_1 = \|f\|_1$, $\|\tilde{g}\|_1 = \|g\|_1$, so the Lemma 1 is proved. Now we are able to prove that

(7) $$\left\|h_0 - \frac{1}{2}\right\| \leqslant 1.$$

(From (6) it follows that this is equivalent to (iii)). If $h_0 = 1$ it holds. Otherwise (6) implies that $\|h_0 - 1\| \geqslant 1$.

Let F_n, G_n, V_n be the following sets

$$F_n = \{f_m^n : m = 1, \ldots, n\},$$
$$G_n = \{g_{m,j}^n m = 1, \ldots, n; j = 1, \ldots, n+1\}$$
$$V_n = \bigcup_{k=1}^{n} F_k \cup G_k.$$

Given a positive number δ. We choose $K \in N$ such that:

(8) $$\|h_0 - \sum_{n=1}^{N} h_{\sigma(n)}\| \leqslant \delta \quad \text{for every } N \geqslant K$$

and for every $m > l > K$

(9) $$\|\sum_{n=l}^{m} h_{\sigma(n)}\| \leqslant \delta$$

$$\sum_{n=1}^{K} h_{\sigma(n)} \text{ will be denoted by } h.$$

Let $M \in N$ be any number such that

$$h_{\sigma(n)} \in V_M \cup F_{M+1} \quad \text{for } n \leqslant K.$$

We define functions h_n^*, $\bar{h}_n$, h^*

$$h_n^* = \begin{cases} h_{\sigma(n)} & \text{if } h_{\sigma(n)} \in V_M \cup F_{M+1} \text{ and } n > K \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{h}_n = \begin{cases} h_{\sigma(n)} & \text{if } h_{\sigma(n)} \in G_{M+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h^* = \sum_{n=K+1}^{\infty} h_n^*.$$

From (3) it follows that $h + h^* = 1$. Thus we have

(10) $$\|h^*\| = \|h-1\| \geqslant \|h_0 - 1\| - \|h_0 - h\| \geqslant 1 - \delta.$$

Let $l_0 = K$ and

(11) $$l_{j+1} = \min\left\{l : \frac{1}{4} - \frac{5}{4}\delta \leqslant \|\sum_{n>l_j}^{l} h_n^*\| \leqslant \frac{1}{4} - \frac{\delta}{4}\right\} \qquad j = 0, 1, 2, 3.$$

(9), (10) justify the above definitions – ((9) implies that $\|h_{\sigma(n)}\| < \delta$, for $n > K$).

We define

$$h^{**}_{j+1} = \sum_{l_j+1}^{l_{j+1}} h^*_n \qquad j = 0, 1, 2, 3$$

$$\bar{h}_{j+1} = \sum_{l_j+1}^{l_{j+1}} \bar{h}_n \qquad j = 0, 1, 2, 3$$

$$\hat{h}_{j+1} = \sum_{l_j+1}^{l_{j+1}} h_{\sigma(n)} \qquad j = 0, 1, 2, 3$$

$$r_j = \hat{h}_j - h^{**}_j - \bar{h}_j \qquad j = 1, 2, 3, 4$$

$$h^{**}_5 = \sum_{l_4+1}^{\infty} h^*_n.$$

Let us remark that r_j, $j = 1, 2, 3, 4$ is a sum of all $h_{\sigma(n)}$ satisfying conditions:

a) $l_j \geqslant n > l_{j-1}$, b) $h_{\sigma(n)} \notin V_M \cup F_{M+1} \cup G_{M+1}$.

Applying Lemma 1 to h^{**}_j and r_j (they depend on different coordinates!) we obtain:

(12) $$\|h^{**}_j + r_j\| \geqslant \|h^{**}_j\| + \frac{\|r_j\|}{2} \qquad j = 1, 2, 3, 4$$

since h^{**}_j is an integer valued function, condition $\|h^{**}_j\| \leqslant \frac{1}{4}$ implies $\lambda(\operatorname{supp} h^{**}_j) \leqslant \frac{1}{4}$. On the other hand, from (9) we have

(13) $$\|\hat{h}_j\| \leqslant \delta$$

so the triangle inequality gives

$$\|\bar{h}_j\| \geqslant \|h^{**}_j + r_j\| - \delta \geqslant \|h^{**}_j\| - \delta \geqslant \frac{1}{4} - \frac{9}{4}\delta.$$

Let us assume that $\|h^{**}_5\| > 11\delta$. In this case we find l_5 such that for

$$\hat{h}^{**}_5 = \sum_{l_4+1}^{l_5} h^*_n \text{ we have } 10\delta < \|\hat{h}^{**}_5\| \leqslant 11\delta.$$

We put $\bar{h}_5 = \sum_{l_4+1}^{l_5} \bar{h}_n$ and similarly as above prove that $\|\bar{h}_5\| \geqslant \|\hat{h}^{**}_5\| - \delta > 9\delta$. However, $1 \geqslant \sum_{n=1}^{\infty} \|\bar{h}_n\| \geqslant \sum_{j=1}^{5} \|\bar{h}_j\| > 1 - 9\delta + 9\delta = 1$, which is a contradiction. Thus $\|h^{**}_5\| \leqslant 11\delta$. We have

$$1 \geqslant \sum_{j=1}^{4} \|\bar{h}_j\| = \sum_{j=1}^{4} \|\hat{h}_j - h^{**}_j - r_j\| \geqslant \sum_{j=1}^{4} \|h^{**}_j + r_j\| - \sum_{j=1}^{4} \|\hat{h}_j\|.$$

Using (12), (13) and (11) we get

$$1 \geqslant 1-5\delta-4\delta+\frac{1}{2}\sum_{j=1}^{4}\|r_j\|,$$

so

$$\sum_{j=1}^{4}\|r_j\| \leqslant 18\delta.$$

Let $H = \sum_{n=1}^{l_4} h_{\sigma(n)} = h+\sum_{j=1}^{4} h_j^{**}+\sum_{j=1}^{4}\bar{h}_j+\sum_{j=1}^{4} r_j$. We have

$$\|H-\tfrac{1}{2}\| = \Big\|h+\sum_{j=1}^{4} h_j^{**}+h_5^{**}+\sum_{j=1}^{4}\bar{h}_j+\sum_{j=1}^{4} r_j-h_5^{**}-\frac{1}{2}\Big\| \tag{14}$$

$$= \Big\|h+h^*-\frac{1}{2}+\sum_{j=1}^{4}\bar{h}_j+\sum_{j=1}^{4} r_j-h_5^{**}\Big\| \leqslant \Big\|\frac{1}{2}+\sum_{j=1}^{4}\bar{h}_j\Big\|$$

$$+\Big\|\sum_{j=1}^{4} r_j\Big\|+\|h_5^{**}\| \leqslant \frac{1}{2}+18\delta+11\delta = \frac{1}{2}+29\delta.$$

Finally (14) and (9) give

$$\|h_0-1/2\| \leqslant \|H-1/2\|+\|h_0-H\| \leqslant 1/2+30\delta.$$

Since δ was an arbitrary positive number we obtain the required result. To prove the Theorem we will apply the method of B. M. Kadec, introduced in [3] and developed in [4].

LEMMA 2 (see [3], in general form in [4]). *Every infinite dimensional Banach space X contains a basic sequence $(e_k)_{k=1}^{\infty}$ such that*

$$\forall_{(t_k)_{k=1}^{\infty}},\ t_k\in \mathbf{R} \quad \Big(\sum_{k=1}^{N}|t_k|^2\Big)^{\frac{1}{2}} \leqslant \Big\|\sum_{k=1}^{N} t_k e_k\Big\| \leqslant (12+\log N)\Big(\sum_{k=1}^{N}|t_k|^2\Big)^{\frac{1}{2}}.$$

Proof—see [3].

Now in $L_2(Q)$ we find an orthogonal system $(s_k)_{k=1}^{\infty}$ such that

$$\forall_n\ h_n,\ h_{\pi(n)}\in \operatorname{span}\{s_1,\ldots, s_{2n}\} \tag{15}$$

and $s_1 = h_1 = 1$. For $Y = \operatorname{span}(e_k)_{k=1}^{\infty}$ we define $T\colon Y\to \overline{\operatorname{span}\{s_1,\ldots, s_k\ldots\}}$ by the formula $T\big(\sum_{k=1}^{\infty} t_k e_k\big) = \sum_{k=1}^{\infty} t_k s_k$.

It follows from Lemma 2 that T is continuous and injective. By virtue of (15) $y_n = T^{-1}(h_n)$ exists, and

$$\sum_{k=1}^{N} y_k,\ \sum_{k=1}^{N} y_{\pi(k)}\in \operatorname{span}\{e_1,\ldots, e_{2N}\}$$

hence

$$\|\sum_{k=1}^{N} y_k\| \leqslant (12+\log 2N)\|T(\sum_{k=1}^{N} y_k)\|_2$$

$$\leqslant (12+\log 2N)\|\sum_{k=1}^{N} h_k\|_2 \leqslant C_0(12+\log 2N)N^{-1/6}.$$

Analogously

$$\|\sum_{k=1}^{N} y_k - y_1\| \leqslant C_1(12+\log 2N)N^{-1/6}.$$

Thus

$$\sum_{n=1}^{\infty} y_n = 0, \quad \sum_{n=1}^{\infty} y_{\pi(n)} = y_1 = e_1.$$

If $\sum_{n=1}^{\infty} y_{\sigma(n)} = y$ for some permutation σ then $T(\sum_{n=1}^{\infty} y_{\sigma(n)}) = T(y)$. However $T(\sum_{n=1}^{\infty} y_{\sigma(n)}) = \sum_{n=1}^{\infty} h_{\sigma(n)}$, hence $T(y) = 0$ or $T(y) = 1$. Since T is injective $y = 0$ or $y = e_1$. So the sequence $(y_n)_{n=1}^{\infty}$ satisfies all conditions of the Theorem.

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М. И. Кадец, К. Возьняковский, **О сериях, перестановки которых обладают лишь двумя суммами**

Цель этой работы — доказать, что в любом банаховом пространстве бесконечной размерности существует последовательность (x_n) такая, что:

$$\operatorname{card}\{x \in X : \exists_{\pi:N\to N},\ x = \sum x_{\pi(n)}\} = 2$$

где π — перестановка множества натуральных чисел.